

RECEIVED: March 25, 2016

REVISED: June 1, 2016

ACCEPTED: June 17, 2016

PUBLISHED: June 27, 2016

# Double supergeometry

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**Martin Cederwall**

*Division for Theoretical Physics, Department of Physics, Chalmers University of Technology,  
SE 412 96 Gothenburg, Sweden*

*E-mail:* [martin.cederwall@chalmers.se](mailto:martin.cederwall@chalmers.se)

**ABSTRACT:** A geometry of superspace corresponding to double field theory is developed, with type II supergravity in  $D = 10$  as the main example. The formalism is based on an orthosymplectic extension  $\text{OSp}(d, d|2s)$  of the continuous T-duality group. Covariance under generalised super-diffeomorphisms is manifest. Ordinary superspace is obtained as a solution of the orthosymplectic section condition. A systematic study of curved superspace Bianchi identities is performed, and a relation to a double pure spinor superfield cohomology is established. A Ramond-Ramond superfield is constructed as an infinite-dimensional orthosymplectic spinor. Such objects in minimal orbits under the  $\text{OSp}$  supergroup (“pure spinors”) define super-sections.

**KEYWORDS:** Space-Time Symmetries, Supergravity Models, Superspaces, Supersymmetry and Duality

**ARXIV EPRINT:** [1603.04684](https://arxiv.org/abs/1603.04684)

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## 1 Introduction

There is by now a significant bulk of work on double geometry [1–26] and exceptional geometry [27–45]. Although some of the work concerns supersymmetry, little attention has been given to supergeometric formulations. In double geometry there are a few papers by Siegel and by Hatsuda et al. [1–3], and in the exceptional setting nothing, to the best of our knowledge. A few papers [45–47] deal with particles and strings in flat superspace. The purpose of the present paper is to investigate double supergeometry, starting from diffeomorphisms on a double superspace. The generalised diffeomorphisms and the corresponding local supersymmetry will thus be manifest in the formalism.

There are several obvious motivations for trying to achieve a formulation where these symmetries are manifest. The foremost may be the belief that a formulation with as much symmetry as possible manifested should be more elegant and perhaps simpler, but there are also more practical issues like the construction of terms in supergravity effective actions restricted both by maximal supersymmetry and by duality.

Our approach closely parallels the geometric formulation of double geometry [18], and provides a natural supersymmetric counterpart. As will be argued, a doubled superspace indeed has twice the number of coordinates, both bosonic and fermionic, as an ordinary superspace. The latter will arise as a solution of the supersymmetric section condition. The rôle of the group  $O(d, d)$  in double geometry is subsumed by an orthosymplectic group

$\text{OSp}(d, d|2s)$ . Unlike previous work on double supergeometry [1–3], our formalism does not use any input from string theory in terms of world-sheet algebras. Our superspace is also considerably smaller, and uses a locally realised symmetry (some real form of)  $\text{Spin}(d) \times \text{Spin}(d)$ , which ties more directly both to double geometry and to ordinary superspace.

First, a review of double geometry will be given in section 2. It will provide the necessary tools to extend to double supergeometry in section 3. There, it will be argued that the Bianchi identities take the theory on shell (in the case of maximal supersymmetry). Support is obtained from the interpretation in terms of a double pure spinor superfield in section 4. Section 5 contains a double superspace formulation of the Ramond-Ramond fields. The structures examined there provide some clues to what will happen in an exceptional setting; this is discussed in the conclusions in section 6, where a further comparison to earlier work also is made.

## 2 Background — double geometry

The geometric formulation of double field theory [18] (see also refs. [7, 13, 16]) is well known. It will be briefly recapitulated here, since most of the calculations in double supergeometry that will be performed in section 3 closely parallel the ones in the bosonic situation. In fact, much of the information will be obtained fairly directly by replacing (anti-)symmetrisations with graded versions.

Let the coordinate basis tangent vectors of doubled space carry an index  $\dot{m}$ . The coordinates  $X^{\dot{m}}$  are the bosonic part of the superspace coordinates of the following section.<sup>1</sup>

The tangent space of doubled space, which is identical to the generalised/doubled tangent space of ordinary space in generalised geometry, is equipped with an  $O(d, d)$  structure defined by an  $O(d, d)$ -invariant metric  $\eta_{\dot{m}\dot{n}}$ . In a suitable basis, where  $dX^{\dot{m}} = (dx^m, d\tilde{x}_m)$  it takes the form

$$\eta_{\dot{m}, \dot{n}} = \begin{pmatrix} 0 & \delta_m^n \\ \delta^m_n & 0 \end{pmatrix}. \quad (2.1)$$

The section condition, the solutions of which locally reduces the dependence of fields on coordinates to that of an ordinary  $d$ -dimensional space, reads

$$\eta^{\dot{m}\dot{n}} \partial_{\dot{m}} \otimes \partial_{\dot{n}} = 0. \quad (2.2)$$

The meaning of “ $\otimes$ ” is that the two derivatives can act on any fields. This means that one looks for a linear subspace of momenta, where all vectors mutually satisfy (2.2). Modulo the choice of basis for  $O(d, d)$ , the solution is given by  $\frac{\partial}{\partial \tilde{x}_m} = 0$ , so fields locally depend only on  $x^m$ .

Generalised diffeomorphisms with a parameter  $\xi^{\dot{m}}$  transform (co)vectors according with the generalised Lie derivative (the Dorfman bracket)

$$\mathcal{L}_\xi V_{\dot{m}} = \xi V_{\dot{m}} + (a - a^T)_{\dot{m}}^{\dot{n}} V_{\dot{n}}, \quad (2.3)$$

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<sup>1</sup>The standard notation is to use an index  $M$ , but this will be reserved for another set of superspace coordinates. The somewhat awkward index convention is consequence of the need of a large number of alphabets later in the paper.

where  $\xi = \xi^{\dot{m}} \partial_{\dot{m}}$  and  $a_{\dot{m}}^{\dot{n}} = \partial_{\dot{m}} \xi^{\dot{n}}$ , and where  $a^T = \eta a^t \eta^{-1}$ . The transformations commute to

$$[\mathcal{L}_\xi, \mathcal{L}_\eta] = \mathcal{L}_{[\xi, \eta]}, \quad (2.4)$$

where

$$[\xi, \eta] = \frac{1}{2}(\mathcal{L}_\xi \eta - \mathcal{L}_\eta \xi). \quad (2.5)$$

is the Courant bracket. The Courant bracket does not satisfy a Jacobi identity, but the violation is a parameter of the form  $\zeta^{\dot{m}} = \eta^{\dot{m}\dot{n}} \partial_{\dot{n}} \lambda$  with  $\mathcal{L}_\zeta = 0$ , representing the singlet reducibility of the gauge transformation (directly inherited from the second order, gauge for gauge, transformation for the  $B$  field).

The generalised geometric fields, the metric and the  $B$ -field, are encoded in a generalised metric or a generalised vielbein. Here, as is normal for superspace, the latter is chosen. The vielbein  $E_{\dot{m}}^{\dot{a}}$  is a group element of  $O(d, d)$ . It is demanded to be covariantly constant when transported by a covariant derivative with generalised affine and spin connections,

$$D_{\dot{m}} E_{\dot{n}}^{\dot{a}} = \partial_{\dot{m}} E_{\dot{n}}^{\dot{a}} + \Gamma_{\dot{m}\dot{n}}^{\dot{p}} E_{\dot{p}}^{\dot{a}} - E_{\dot{n}}^{\dot{b}} \Omega_{\dot{m}\dot{b}}^{\dot{a}} = 0. \quad (2.6)$$

Covariance of the covariant derivative dictates that  $\Gamma$  transforms inhomogeneously under generalised diffeomorphisms:

$$\delta_\xi \Gamma_{\dot{m}\dot{n}}^{\dot{p}} = \mathcal{L}_\xi \Gamma_{\dot{m}\dot{n}}^{\dot{p}} - \partial_{\dot{m}}(\partial_{\dot{n}} \xi^{\dot{p}} - \partial^{\dot{p}} \xi_{\dot{n}}). \quad (2.7)$$

This implies that the totally antisymmetric part  $\Gamma_{[\dot{m}\dot{n}\dot{p}]}$  transforms covariantly. It is defined as torsion,<sup>2</sup>

$$T_{\dot{m}\dot{n}\dot{p}} = -\frac{3}{2} \Gamma_{[\dot{m}\dot{n}\dot{p}]}. \quad (2.8)$$

Next, curvature is constructed. This can be done as in ref. [18], by considering combinations of  $\partial\Gamma$  and  $\Gamma^2$  for which the inhomogeneous transformations cancel. Another possibility is to use the compatibility equation (2.6). Then one will also get information about how the curvature is expressed in terms of the spin connection. The idea is the standard one: by taking derivatives of the compatibility equation, one searches for an equality between two expressions, of which one is manifestly covariant under transformations in the locally realised subalgebra  $so(d) \oplus so(d)$ , and the other is a tensor. Then one concludes that the two expressions, which are the expressions for curvature in terms of affine connection and spin connection, respectively, are equal and share both covariance properties.

Taking one derivative on eq. (2.6), antisymmetrising, and replacing the derivatives on the vielbein leads to

$$(\partial_{[\dot{m}} \Gamma_{\dot{n}]} + \Gamma_{[\dot{m}} \Gamma_{\dot{n}]})_{\dot{p}}^{\dot{q}} E_{\dot{q}}^{\dot{a}} - E_{\dot{p}}^{\dot{b}} (\partial_{[\dot{m}} \Omega_{\dot{n}]} + \Omega_{[\dot{m}} \Omega_{\dot{n}]})_{\dot{b}}^{\dot{a}} = 0. \quad (2.9)$$

This is precisely the step taken in ordinary geometry to derive the Riemann tensor. What fails in double geometry is that  $\partial_{[\dot{m}} \Omega_{\dot{n}]}$  is not a tensor, since  $\Gamma_{[\dot{m}\dot{n}]}^{\dot{p}}$  is not torsion. Eq. (2.9)

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<sup>2</sup>This normalisation, which differs from the one in ref. [18], is conventional for later simplification, and has essentially to do with the retraining of the torsion from 2-form to “3-form”.

certainly holds, but can not be used to extract any useful (covariant) information. The derivative on  $\Omega$  can be covariantised, but one needs to compensate with a  $\Gamma\Omega$  term.

In refs. [1, 18], a 4-index curvature was constructed by symmetrisation in the pairs of indices. Explicitly,

$$\begin{aligned}
 & (\partial_{[\dot{m}}\Gamma_{\dot{n}]} + \Gamma_{[\dot{m}}\Gamma_{\dot{n}]})_{\dot{p}\dot{q}} - \frac{1}{4}\Gamma_{\dot{r}\dot{m}\dot{n}}\Gamma^{\dot{r}}_{\dot{p}\dot{q}} \\
 & - E_{\dot{p}}^{\dot{a}}E_{\dot{q}}^{\dot{b}}(D_{[\dot{m}}^{(\Gamma)}\Omega_{\dot{n}]} + \Omega_{[\dot{m}}\Omega_{\dot{n}]})_{\dot{a}\dot{b}} \\
 & + E_{\dot{p}}^{\dot{a}}E_{\dot{q}}^{\dot{b}}\Gamma_{[\dot{m}\dot{n}]}^{\dot{r}}\Omega_{\dot{r}\dot{a}\dot{b}} + \frac{1}{4}\Gamma_{\dot{r}\dot{m}\dot{n}}\Gamma^{\dot{r}}_{\dot{p}\dot{q}} \\
 & + (\dot{m}\dot{n} \leftrightarrow \dot{p}\dot{q}) = 0.
 \end{aligned} \tag{2.10}$$

The combination of connections on the third line turns out to be a tensor under generalised diffeomorphisms. If the vielbeins, for simplicity, are suppressed, the compatibility equation tells us that  $\Gamma_{\dot{m}} - \Omega_{\dot{m}}$  carries a derivative with index  $\dot{m}$ . Therefore, the section condition can be used to obtain

$$\begin{aligned}
 0 &= \frac{1}{2}(-\Gamma_{\dot{r}} + \Omega_{\dot{r}})_{\dot{m}\dot{n}}(-\Gamma^{\dot{r}} + \Omega^{\dot{r}})_{\dot{p}\dot{q}} \\
 &= \frac{1}{4}\Gamma_{\dot{r}\dot{m}\dot{n}}\Gamma^{\dot{r}}_{\dot{p}\dot{q}} + (T_{\dot{m}\dot{n}\dot{r}} + \Gamma_{[\dot{m}\dot{n}]\dot{r}})\Omega_{\dot{r}\dot{p}\dot{q}} + \frac{1}{4}\Omega_{\dot{r}\dot{m}\dot{n}}\Omega^{\dot{r}}_{\dot{p}\dot{q}} + (\dot{m}\dot{n} \leftrightarrow \dot{p}\dot{q}).
 \end{aligned} \tag{2.11}$$

The only non-tensorial terms (under generalised diffeomorphisms) are the ones containing  $\Gamma$ , and match the ones on the third line of eq. (2.10). On the other hand, the first line in eq. (2.10) is manifestly invariant under the local subgroup. This means that there is a curvature tensor:

$$\begin{aligned}
 R_{\dot{m}\dot{n}\dot{p}\dot{q}} &= (\partial_{[\dot{m}}\Gamma_{\dot{n}]} + \Gamma_{[\dot{m}}\Gamma_{\dot{n}]})_{\dot{p}\dot{q}} - \frac{1}{4}\Gamma_{\dot{r}\dot{m}\dot{n}}\Gamma^{\dot{r}}_{\dot{p}\dot{q}} + (\dot{m}\dot{n} \leftrightarrow \dot{p}\dot{q}) \\
 &= E_{\dot{p}}^{\dot{a}}E_{\dot{q}}^{\dot{b}}(D_{[\dot{m}}^{(\Gamma)}\Omega_{\dot{n}]} + \Omega_{[\dot{m}}\Omega_{\dot{n}]} + T_{\dot{m}\dot{n}}^{\dot{r}}\Omega_{\dot{r}})_{\dot{a}\dot{b}} \\
 &+ \frac{1}{4}E_{\dot{m}}^{\dot{a}}E_{\dot{n}}^{\dot{b}}E_{\dot{p}}^{\dot{c}}E_{\dot{q}}^{\dot{d}}\Omega_{\dot{r}\dot{a}\dot{b}}\Omega^{\dot{r}}_{\dot{c}\dot{d}} + (\dot{m}\dot{n} \leftrightarrow \dot{p}\dot{q}).
 \end{aligned} \tag{2.12}$$

By construction, the curvature is antisymmetric in  $[\dot{m}\dot{n}]$  and in  $[\dot{p}\dot{q}]$ , and symmetric under interchange of the pairs of indices. The completely antisymmetric part figures in the torsion Bianchi identity, see below.

The second form of  $R$  is maybe not very useful compared to the first one. However, it conveys one very important piece of information, which is not manifest from its expression in the affine connection, namely, that it only gets contributions from terms where at least one of the two pairs takes values in  $so(d) \oplus so(d)$ . The corresponding property of the curvature on superspace will be relied upon when the superspace Bianchi identities are investigated.

In addition to the generalised metric, a generalised dilaton  $\Phi$  is introduced. Its transformation is given by

$$\delta_{\xi}e^{-2\Phi} = \partial_{\dot{m}}(\xi^{\dot{m}}e^{-2\Phi}), \tag{2.13}$$

which means that  $e^{-2\Phi}$  is a density with weight  $w = 1$ . The action of the covariant derivative  $D_{\dot{m}}$  on a density contains an extra term  $w\Gamma_{\dot{m}\dot{n}}^{\dot{n}}\psi$  for any tensor density  $\psi$ .

Clearly,  $e^{2\Phi} D_{\dot{m}} e^{-2\Phi}$  is a vector, and it can be seen as an additional part of torsion,  $T_{\dot{m}}$ . It can be used (with knowledge of the dilaton field) to determine a vector part of the connection,

$$\Gamma_{\dot{m}\dot{m}}^{\dot{n}} = 2\partial_{\dot{m}}\Phi + T_{\dot{m}}^{\dot{n}}. \quad (2.14)$$

Unlike in ordinary geometry, the compatibility of the vielbein together with specification of the torsion is not sufficient to determine the connections completely. There are certain  $so(d) \oplus so(d)$ -modules, where torsion is absent, and eq. (2.6) only gives information about  $\Gamma - \Omega$ . These modules are the irreducible hooks,  $\boxplus$ , under the two  $so(d)$ 's. Any physical relations, such as the equations of motion for the generalised vielbein, must avoid using the undefined connections. This turns out to be the case when they are formally derived by variation of the pseudo-action

$$S = \int d^{2d}X e^{-2\Phi} R, \quad (2.15)$$

$R$  being the scalar curvature obtained from the curvature in eq. (2.12). See ref. [18] for details.

When later superspace geometry is investigated, torsion will be non-vanishing, and we need the torsion Bianchi identity. A direct calculation leads to the identity<sup>3</sup>

$$4D_{[\dot{m}}T_{\dot{n}\dot{p}\dot{q}]} + 6T_{[\dot{m}\dot{n}}^{\dot{r}}T_{\dot{p}\dot{q}]\dot{r}} = -3R_{[\dot{m}\dot{n}\dot{p}\dot{q}]} . \quad (2.16)$$

### 3 Double supergeometry

The vielbein of double of exceptional geometry is restricted to be a group element of the duality group  $O(d, d)$  or  $E_{n(n)} \times \mathbb{R}^+$ . If supersymmetry is to be made manifest in terms of superfields, it seems necessary that superspace, even in the flat case, has an interpretation as supergeometry. It does not seem consistent with super-diffeomorphisms to constrain a bosonic corner of a super-vielbein, or even to consider a bosonic part of a vector to transform under a restricted subgroup. A direct solution would entail a super-extension of the duality group. For double geometry, the T-duality groups allow for natural extensions in the form of supergroups  $OSp(d, d|2s)$ .

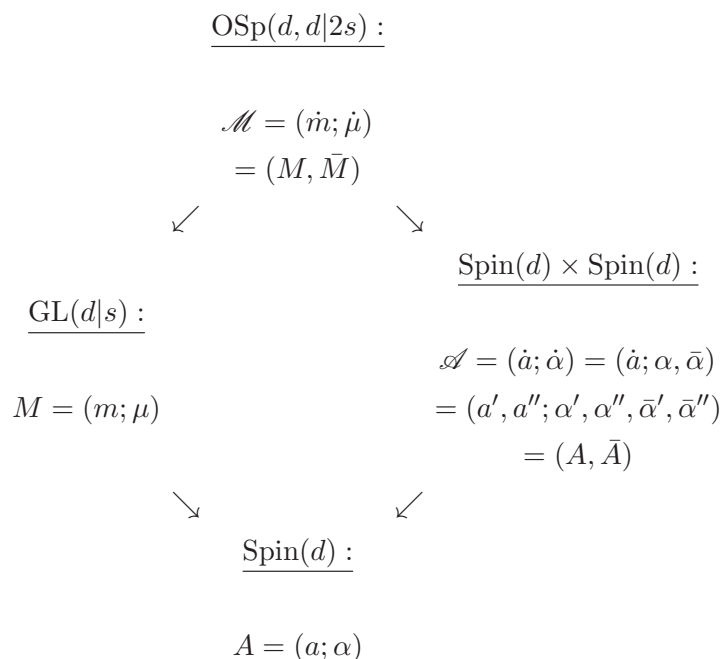
This may naïvely seem to rule out supergeometry based on exceptional U-duality groups, due to the “non-existence” of exceptional supergroups. We will come back to this issue in the conclusions of section 6, as it turns out that the behaviour of the Ramond-Ramond fields in double supergeometry points strongly towards infinite-dimensional supergroups in the exceptional cases.

#### 3.1 Notation and OSp basics

In section 3.2, generalised super-diffeomorphisms on a  $(2d|2s)$ -dimensional superspace will be defined. The coordinate differentials form the fundamental module of the orthosymplectic supergroup  $OSp(d, d|2s)$ , and this structure will be preserved by generalised super-diffeomorphisms, in complete analogy to the  $O(d, d)$  structure in double geometry.

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<sup>3</sup>The purpose of the normalisation of the torsion tensor in eq. (2.8) was to obtain natural coefficients here ( $4 = \frac{4!}{3!}$ ,  $6 = \frac{4!}{(2!)^2}$ ).



**Figure 1.** Summary of index notation for the various relevant groups and supergroups.

The orthosymplectic group allows an invariant metric  $H_{\mathcal{MN}}$  on superspace.<sup>4</sup> A basis may be chosen so that

$$H_{\mathcal{MN}} = \begin{pmatrix} \eta_{mn} & 0 \\ 0 & \epsilon_{\dot{\mu}\dot{\nu}} \end{pmatrix}, \quad (3.1)$$

where  $\eta$  is the symmetric  $O(d, d)$  metric, and  $\epsilon$  is the antisymmetric invariant tensor of  $\text{Sp}(2s)$ . Thus, tangent indices  $\mathcal{M} = (\dot{m}; \dot{\mu})$  are used in coordinate basis. Later, when a generalised super-vielbein is introduced, there will also be flat indices. The locally realised subgroup (analogous to the Lorentz group in ordinary superspace) will be a real form of  $\text{Spin}(d) \times \text{Spin}(d)$ ,<sup>5</sup> and the corresponding indices are denoted  $\mathcal{A} = (\dot{a}; \dot{\alpha}) = (a', a''; \alpha', \alpha'')$ . Here the dotted indices are the collective  $2d$  and  $2s$  flat indices, which are vectors and spinors, respectively, under the two factors of  $\text{Spin}(d)$ . Vectors and spinors under the first and second  $\text{Spin}(d)$  are denoted with primed and double-primed indices (this refers to type II, which will be our main interest).

<sup>4</sup>It seems quite clear that it may be possible to introduced a arbitrary, possibly curved, superspace metric instead of  $H$ , along the lines of refs. [25, 26]. That option is not explored here, but a constant, algebraic,  $H$  is used.

<sup>5</sup>There are certainly other possible choices for locally realised subgroups or even sub-supergroups. However, one important rôle of the choice of local subgroup is to turn fermions into spinors. This is why, in ordinary supergeometry, the local subgroup of  $\text{GL}(d|s)$  is taken as the Lorentz group, and  $s$  needs to coincide with the dimensionality of some spinor representation. The choice made here reproduces the choice of local subgroup in ordinary supergeometry; see figure 1.

Another natural decomposition of  $\mathcal{M}$  is in terms of a  $\mathrm{GL}(d|s) \subset \mathrm{OSp}(d, d|2s)$  subgroup. Then  $dZ^{\mathcal{M}} = (dX^{\dot{m}}; d\Theta^{\dot{\mu}}) = (dx^m, d\tilde{x}_m; d\theta^\mu, d\tilde{\theta}_\mu)$ , and the invariant metric takes the form

$$H_{\mathcal{M}\mathcal{N}} = \begin{pmatrix} 0 & \delta_m^{\dot{n}} & 0 & 0 \\ \delta^{\dot{m}}_n & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta_\mu^{\dot{\nu}} \\ 0 & 0 & -\delta^\mu_{\dot{\nu}} & 0 \end{pmatrix}. \quad (3.2)$$

This decomposition is relevant for local solutions to the strong section condition (see below). Occasionally, the indices  $dz^M = (dx^m, d\theta^\mu)$  and  $d\tilde{z}_M = (d\tilde{x}_m, d\tilde{\theta}_\mu)$  are used. The coordinates  $z^M$  span an ordinary superspace. The corresponding flat indices are written  $A = (a, \alpha)$ , a vector and spinor under the diagonal subgroup  $\mathrm{Spin}(d)$  which is also a subgroup of  $\mathrm{GL}(d|s)$ . The index conventions are summarised in the diagram of figure 1.

Transformation matrices will belong to the adjoint of the Lie superalgebra  $\mathfrak{osp}(d, d|2s)$ . In section 3.2, they will be formed from the derivative of a super-diffeomorphism parameter, i.e., from the matrix  $a_{\mathcal{M}\mathcal{N}} = \partial_{\mathcal{M}}\xi^{\mathcal{N}}$ . Therefore, there is a need to consistently multiply matrices, matrices and vectors, raise and lower indices and perform transpositions. The last operation is essential in order to form the adjoint (graded antisymmetric) from an arbitrary matrix.

Since vectors and matrices contain bosonic and fermionic objects, ordering is important. A convention is used where covectors and vectors transform as

$$\begin{aligned} \delta_f W_{\mathcal{M}} &= f_{\mathcal{M}}^{\mathcal{N}} W_{\mathcal{N}}, \\ \delta_f V^{\mathcal{M}} &= -V^{\mathcal{N}} f_{\mathcal{N}}^{\mathcal{M}} \end{aligned} \quad (3.3)$$

under an  $\mathfrak{osp}(d, d|2s)$ -transformation with an algebra element  $f$  (more about it later). Then  $V^{\mathcal{M}} W_{\mathcal{M}}$  (in this order) is invariant. Indices are always contracted in the direction NW-SE, and contraction is performed on neighbouring indices.

With these conventions, indices are lowered by right multiplication with  $H$  and raised by left multiplication by a matrix  $\hat{H}$ . This is not the inverse of  $H$ , but

$$\hat{H}^{\mathcal{M}\mathcal{N}} = \begin{pmatrix} \eta^{\dot{m}\dot{n}} & 0 \\ 0 & \epsilon^{\dot{\mu}\dot{\nu}} \end{pmatrix} \quad (3.4)$$

(taking the inverse of a matrix with this index structure is not allowed by the conventions).

This also introduces a consistency check for the conventions for raising and lowering of indices, since vectors and covectors form the same module. Define  $V^{\mathcal{M}} = \hat{H}^{\mathcal{M}\mathcal{N}} V_{\mathcal{N}}$ . All objects dealt with have fermion number equal to the number of “fermionic” indices, i.e., fundamental  $\mathrm{Sp}(2s)$  indices. Fermion number is denoted  $\varepsilon(\mathcal{M})$ , which takes the value 0 for  $\mathcal{M} = \dot{m}$  and 1 for  $\mathcal{M} = \dot{\mu}$ . The invariant metric is graded symmetric:  $\hat{H}^{\mathcal{N}\mathcal{M}} = (-1)^{\varepsilon(\mathcal{M})\varepsilon(\mathcal{N})} \hat{H}^{\mathcal{M}\mathcal{N}}$ . It is convenient to define a graded transpose as  $(M^T)^{\mathcal{M}\mathcal{N}} = (-1)^{\varepsilon(\mathcal{M})\varepsilon(\mathcal{N})} M^{\mathcal{N}\mathcal{M}}$ .



Checking the invariance of  $V^{\mathcal{M}}W_{\mathcal{M}} = \hat{H}^{\mathcal{M}\mathcal{N}}V_{\mathcal{N}}W_{\mathcal{M}}$ , one gets

$$\begin{aligned}\delta_f(V^{\mathcal{M}}W_{\mathcal{M}}) &= \hat{H}^{\mathcal{M}\mathcal{N}}(f_{\mathcal{N}}^{\mathcal{P}}V_{\mathcal{P}}W_{\mathcal{M}} + V_{\mathcal{N}}f_{\mathcal{M}}^{\mathcal{P}}W_{\mathcal{P}}) \\ &= (\hat{H}^{\mathcal{N}\mathcal{P}}f_{\mathcal{P}}^{\mathcal{M}} + (-1)^{\varepsilon(\mathcal{M})\varepsilon(\mathcal{N})}\hat{H}^{\mathcal{M}\mathcal{P}}f_{\mathcal{P}}^{\mathcal{N}})V_{\mathcal{M}}W_{\mathcal{N}} \\ &= (\hat{H}f + (\hat{H}f)^T)^{\mathcal{N}\mathcal{M}}V_{\mathcal{M}}W_{\mathcal{N}},\end{aligned}\quad (3.5)$$

where the sign factor comes from taking  $f$  and  $V$  past each other, and from transposing  $\hat{H}$ . This shows that  $f$ , after its first index is raised with  $\hat{H}$ , must be graded antisymmetric,  $\hat{H}a + (\hat{H}a)^T = 0$ . It is convenient to extend the graded transpose to matrices with the index structure  $(\cdot)^{\mathcal{M}\mathcal{N}}$  by

$$(A^T)^{\mathcal{M}\mathcal{N}} = (H(\hat{H}A)^T)^{\mathcal{M}\mathcal{N}}, \quad (3.6)$$

i.e.,

$$(A^T)^{\mathcal{M}\mathcal{N}} = (-1)^{\varepsilon(\mathcal{M})\varepsilon(\mathcal{N})}(HA^t\hat{H}^t)^{\mathcal{M}\mathcal{N}} = (-1)^{(\varepsilon(\mathcal{M})+\varepsilon(\mathcal{N}))\varepsilon(\mathcal{N})}(HA^t\hat{H})^{\mathcal{M}\mathcal{N}}. \quad (3.7)$$

Then, the superalgebra element obeys  $f + f^T = 0$ , and given any supermatrix  $a_{\mathcal{M}\mathcal{N}}$ , an adjoint element is formed as  $f = a - a^T$ . Defined this way, the graded transposition is an anti-involution with respect to supermatrix multiplication, and obeys the standard rule

$$(AB)^T = B^T A^T, \quad (3.8)$$

since

$$\begin{aligned}((AB)^T)^{\mathcal{M}\mathcal{N}} &= (-1)^{(\varepsilon(\mathcal{M})+\varepsilon(\mathcal{N}))\varepsilon(\mathcal{N})}(H(AB)^t\hat{H})^{\mathcal{M}\mathcal{N}} \\ &= (-1)^{(\varepsilon(\mathcal{M})+\varepsilon(\mathcal{N}))\varepsilon(\mathcal{N})+(\varepsilon(\mathcal{M})+\varepsilon(\mathcal{P}))(\varepsilon(\mathcal{P})+\varepsilon(\mathcal{N}))}(HB^t\hat{H})^{\mathcal{M}\mathcal{P}}(HA^t\hat{H})^{\mathcal{P}\mathcal{N}} \\ &= (-1)^{(\varepsilon(\mathcal{M})+\varepsilon(\mathcal{P}))\varepsilon(\mathcal{P})+(\varepsilon(\mathcal{P})+\varepsilon(\mathcal{N}))\varepsilon(\mathcal{N})}(HB^t\hat{H})^{\mathcal{M}\mathcal{P}}(HA^t\hat{H})^{\mathcal{P}\mathcal{N}}\end{aligned}\quad (3.9)$$

(this is not a meaningful/covariant statement for other index structures, since then the NW-SE convention is broken). There are also statements like

$$\begin{aligned}W_{\mathcal{M}} &= A_{\mathcal{M}\mathcal{N}}V_{\mathcal{N}} \iff W^{\mathcal{M}} = V^{\mathcal{N}}(A^T)_{\mathcal{N}}^{\mathcal{M}}, \\ A_{\mathcal{M}\mathcal{N}} &= V_{\mathcal{M}}W^{\mathcal{N}} \iff (A^T)^{\mathcal{M}\mathcal{N}} = W_{\mathcal{M}}V^{\mathcal{N}}\end{aligned}\quad (3.10)$$

etc. The conventions lead to formulas free of extra signs due to fermion number.

The scalar part of a matrix  $M_{\mathcal{M}\mathcal{N}}$  sits in its supertrace,

$$\text{Str}M = \text{tr}(\hat{H}MH) = (-1)^{\varepsilon(\mathcal{M})}M_{\mathcal{M}}^{\mathcal{M}} = M_{\dot{m}}^{\dot{m}} - M_{\dot{\mu}}^{\dot{\mu}}, \quad (3.11)$$

where the sign can be seen as a consequence of the NW-SE rule, implemented in the first step. This is because, in general, the trace of a commutator is not zero, but acquires a sign factor due to ordering. The supertrace of a commutator is invariant. Still, the unit matrix  $\delta_{\mathcal{M}\mathcal{N}}$  is of course invariant. The singlet part of a matrix is

$$M_{\mathcal{M}}^{(1)\mathcal{N}} = \frac{1}{2d-2s}\delta_{\mathcal{M}\mathcal{N}}\text{Str}M. \quad (3.12)$$

(If  $d = s$ , the superalgebra  $osp(2d|2d)$  is not simple, but contains an ideal generated by the unit matrix. This will not be relevant to our applications.)

### 3.2 Generalised super-diffeomorphisms

It is a straightforward exercise to define doubled super-diffeomorphisms, where parameters are in the fundamental  $(2d|2s)$ -dimensional module of  $\text{OSp}(d, d|2s)$ . The key point is the section condition.

A solution to the section condition in bosonic double geometry, eq. (2.2), implies that all fields (and gauge parameters) locally are independent of a  $d$ -dimensional isotropic subset of the coordinates. A choice of section breaks  $O(d, d)$  locally to  $\text{GL}(d) \ltimes \Lambda_2$ , where  $\text{GL}(d)$  rotates the physical directions and  $\Lambda_2$  is the linear space of 2-forms. The supergroup provides a covariant version on superspace, namely

$$\hat{H}^{\mathcal{M}\mathcal{N}} \partial_{\mathcal{N}} \otimes \partial_{\mathcal{M}} = 0, \quad (3.13)$$

where  $\hat{H}^{\mathcal{M}\mathcal{N}}$  is the  $\text{OSp}(d, d|2s)$ -invariant metric. Note that the section condition should be formulated on naked derivatives, carrying indices in coordinate basis (“curved indices”).

The section condition (only the “strong” version is considered, necessary for the algebra of generalised super-diffeomorphisms) should be interpreted as a condition on a (maximal) linear subspace of momenta, such that all momenta  $p, p'$  in the subspace satisfy  $\hat{H}^{\mathcal{M}\mathcal{N}} p_{\mathcal{N}} p'_{\mathcal{M}} = 0$ . This amounts to finding a maximal isotropic subspace of (co)tangent vectors. Locally, all fields depend only on the corresponding coordinates. Modulo the choice of  $\text{OSp}$  basis, a solution to the section condition can always locally be brought to the form  $\frac{\partial}{\partial x_m} = 0$ ,  $\frac{\partial}{\partial \theta_\mu} = 0$ . Such a solution breaks  $\text{OSp}(d, d|2s)$  to  $\text{GL}(d|s) \ltimes \Lambda_2$ , where now the first factor provides linear transformations on ordinary superspace and the second one is the space of super-2-forms. Ordinary superspace is (locally) a solution of the section condition on doubled superspace. Solutions of the  $\text{OSp}$  section condition are also parametrised by pure orthosymplectic spinors, in analogy with the bosonic case, as will be described in section 5.

Now, the generalised super-diffeomorphisms take the form

$$\begin{aligned} \mathcal{L}_\xi V^{\mathcal{M}} &= \xi V^{\mathcal{M}} - V^{\mathcal{N}} (a - a^T)_{\mathcal{N}}{}^{\mathcal{M}}, \\ \mathcal{L}_\xi V_{\mathcal{M}} &= \xi V_{\mathcal{M}} + (a - a^T)_{\mathcal{M}}{}^{\mathcal{N}} V_{\mathcal{N}}. \end{aligned} \quad (3.14)$$

where  $\xi = \xi^{\mathcal{M}} \partial_{\mathcal{M}}$  and  $a_{\mathcal{M}}{}^{\mathcal{N}} = \partial_{\mathcal{M}} \xi^{\mathcal{N}}$  (the two expressions for the transformations are of course equivalent). A short calculation using the definitions above, and the section condition on the form  $a^T b = 0$  etc., shows that the commutator of two super-diffeomorphisms give a new super-diffeomorphism:

$$[\mathcal{L}_\xi, \mathcal{L}_\eta] = \mathcal{L}_{[\xi, \eta]}, \quad (3.15)$$

where

$$[\xi, \eta] = \frac{1}{2} (\mathcal{L}_\xi \eta - \mathcal{L}_\eta \xi). \quad (3.16)$$

Just like for the bosonic generalised diffeomorphisms, there is a slight violation of the Jacobi identity, taking the form of a reducibility, related to a trivial parameter  $\zeta_{\mathcal{M}} = \partial_{\mathcal{M}} \lambda$  with  $\mathcal{L}_\zeta = 0$ .

It has thus been shown that it is essential to have the OSp structure already at the level of “curved” (coordinate basis) indices, on which generalised super-diffeomorphisms act. This implies that there must be a doubling not only of the bosonic directions (as compared to physical space), but also of the fermionic ones (as compared to ordinary superspace). Namely, if one wants to attach engineering dimension 1 to any bosonic derivative  $\partial_{\tilde{m}}$ , then it is not consistent that all fermionic derivatives have dimension  $\frac{1}{2}$ . Neither should one expect a formalism leading to a “spinor metric”  $\epsilon$  on ordinary superspace. The fermionic coordinates here consist of  $s$  coordinates  $\theta^\mu$  and  $s$  “extra” coordinates  $\tilde{\theta}_\mu$ . Letting  $\theta$  and  $\tilde{\theta}$  carry dimensions  $-\frac{1}{2}$  and  $-\frac{3}{2}$  respectively is consistent with the OSp structure. Reduction to ordinary superspace is a maximal solution of the section condition on doubled superspace.

Having established the super-diffeomorphisms, it is straightforward to continue with super-vielbeins, connections and curvature. The construction parallels the one in the bosonic case, with the orthogonal group replaced by the orthosymplectic supergroup.

### 3.3 Vielbein, connections and compatibility

An affine connection on superspace should be defined so that

$$D_{\mathcal{M}}V_{\mathcal{N}} = \partial_{\mathcal{M}}V_{\mathcal{N}} + \Gamma_{\mathcal{M}\mathcal{N}}^{\mathcal{P}}V_{\mathcal{P}} \quad (3.17)$$

transforms as a tensor. This forces the connection to transform as

$$\delta_{\xi}\Gamma_{\mathcal{M}\mathcal{N}}^{\mathcal{P}} = \mathcal{L}_{\xi}\Gamma_{\mathcal{M}\mathcal{N}}^{\mathcal{P}} - \partial_{\mathcal{M}}(\partial_{\mathcal{N}}\xi^{\mathcal{P}} - \xi_{\mathcal{N}}^{\mathcal{P}}) + \xi_{\mathcal{N}}^{\mathcal{Q}}\partial_{\mathcal{M}}\Gamma_{\mathcal{Q}}^{\mathcal{P}}. \quad (3.18)$$

The connection should, seen as matrices  $\Gamma_{\mathcal{M}}$ , take values in the Lie superalgebra  $osp(d, d|2s)$ ,  $\Gamma_{\mathcal{M}} + (\Gamma_{\mathcal{M}})^T = 0$ . This is manifestly the case for the inhomogeneous term in eq. (3.18).

The entire connection comes in the tensor product of the fundamental with the graded antisymmetric module. Lower the last index with the invariant metric according to  $\Gamma_{\mathcal{M}\mathcal{N}\mathcal{P}} = \Gamma_{\mathcal{M}\mathcal{N}}^{\mathcal{Q}}H_{\mathcal{Q}\mathcal{P}}$ . It then follows that the totally graded antisymmetric part of the connection transforms as a tensor, which is identified as torsion:

$$T_{\mathcal{M}\mathcal{N}\mathcal{P}} = -\frac{3}{2}\Gamma_{[\mathcal{M}\mathcal{N}\mathcal{P}]} \quad (3.19)$$

A super-vielbein  $E_{\mathcal{M}}^{\mathcal{A}}$  is restricted to be a group element of  $OSp(d, d|2s)$ . Recall that the inertial index  $\mathcal{A} = (\dot{\alpha}; \dot{\alpha}) = (\dot{\alpha}; \alpha, \bar{\alpha})$  labels representations of the locally realised subgroup  $\text{Spin}(d) \times \text{Spin}(d)$ . The inertial index  $\alpha$  will, in a type II theory, label the module  $s \otimes 1 \oplus 1 \otimes s$ , where  $s$  is a spinor module of  $\text{Spin}(d)$ . This represents the dimension- $\frac{1}{2}$  directions, while the conjugate index  $\bar{\alpha}$  represents the dimension- $\frac{3}{2}$  directions.<sup>6</sup> In principle, one may imagine also a chiral situation, with  $S = s \otimes 1$ , and of course also extended supersymmetry with extra R-symmetry group  $R$ . The main example will be type II theory, with  $d = 10$  and  $s = 32$ . Starting from the “structure group”  $OSp(10, 10|64)$ , the locally realised group is  $\text{Spin}(1, 9) \times \text{Spin}(1, 9)$ .

<sup>6</sup> A small sloppiness in index notation has been allowed in the use of the same letter  $\alpha$  for the sum of the spinor modules of the two  $\text{Spin}(d)$ ’s (to the right in figure 1) as for the spinor index of the diagonal subgroup (the bottom of figure 1). There should be no danger in this, since they label the same vector space.

The vielbein is demanded to satisfy a covariant constancy (compatibility) condition,

$$D_{\mathcal{M}} E_{\mathcal{N}}^{\mathcal{A}} = \partial_{\mathcal{M}} E_{\mathcal{N}}^{\mathcal{A}} + \Gamma_{\mathcal{M}\mathcal{N}}^{\mathcal{P}} E_{\mathcal{P}}^{\mathcal{A}} - \Omega_{\mathcal{M}\mathcal{B}}^{\mathcal{A}} E_{\mathcal{N}}^{\mathcal{B}} = 0. \quad (3.20)$$

Here, a spin connection on superspace, taking values in (the relevant real form of)  $so(d) \oplus so(d)$  has been introduced.

At this stage, it should have become obvious that, as long as only curved indices are concerned, all considerations from double geometry can safely be extrapolated to double supergeometry. For example, a 4-index curvature tensor can be formed from the affine super-connection as

$$\begin{aligned} R_{\mathcal{M}\mathcal{N}\mathcal{P}\mathcal{Q}} &= \partial_{[\mathcal{M}} \Gamma_{\mathcal{N}]}^{\mathcal{R}} \Gamma_{\mathcal{P}\mathcal{Q}}^{\mathcal{R}} + (-1)^{\varepsilon(\mathcal{N})\varepsilon(\mathcal{R})} \Gamma_{[\mathcal{M}|\mathcal{P}]}^{\mathcal{R}} \Gamma_{\mathcal{N}]\mathcal{Q}}^{\mathcal{R}} \\ &\quad - \frac{1}{4} (-1)^{(\varepsilon(\mathcal{M})+\varepsilon(\mathcal{N}))\varepsilon(\mathcal{R})} \Gamma_{\mathcal{M}\mathcal{N}}^{\mathcal{R}} \Gamma_{\mathcal{P}\mathcal{Q}}^{\mathcal{R}} \\ &\quad + (-1)^{(\varepsilon(\mathcal{M})+\varepsilon(\mathcal{N}))(\varepsilon(\mathcal{P})+\varepsilon(\mathcal{Q}))} (\mathcal{M}\mathcal{N} \leftrightarrow \mathcal{P}\mathcal{Q}) \end{aligned} \quad (3.21)$$

(The awkward sign factors are simply a consequence of the impossibility to write the contractions in terms of neighbouring indices). There is also an expression in terms of  $\Omega$ , reflecting eq. (2.12) in double geometry. It is not restated here. It gives again the important information that at least one of the pairs of indices on the curvature  $R_{\mathcal{A}\mathcal{B}\mathcal{C}\mathcal{D}}$  with flat indices takes values in the locally realised algebra  $so(d) \oplus so(d)$ . This will be used in section 3.4.

The super-torsion Bianchi identity reads

$$4D_{[\mathcal{M}} T_{\mathcal{N}\mathcal{P}\mathcal{Q}}] + 6T_{[\mathcal{M}\mathcal{N}}^{\mathcal{R}} T_{\mathcal{P}\mathcal{Q}}^{\mathcal{R}}] = -3R_{[\mathcal{M}\mathcal{N}\mathcal{P}\mathcal{Q}].} \quad (3.22)$$

A generalised dilaton superfield with weight 1 is introduced just as in the bosonic case. It transforms as

$$\delta_{\xi} e^{-2\Phi} = \partial^{\mathcal{M}} (\xi_{\mathcal{M}} e^{-2\Phi}). \quad (3.23)$$

A tensor density  $\Psi$  with weight  $w$  transforms as

$$\delta_{\xi} \Psi = \mathcal{L}_{\xi}^{(w)} \Psi = \mathcal{L}_{\xi} \Psi + w \partial^{\mathcal{M}} \xi_{\mathcal{M}} \Psi, \quad (3.24)$$

and its covariant derivative is

$$D_{\mathcal{M}}^{(w)} \Psi = D_{\mathcal{M}} \Psi - w \Gamma_{\mathcal{N}\mathcal{M}}^{\mathcal{N}} \Psi. \quad (3.25)$$

There are however some differences when it comes to the interplay between the OSp group and its locally realised subgroup. The local subgroup is the same as in the bosonic case (but with a different action, of course, the fermions becoming spinors), while the torsion is considerably larger. It is straightforward to check which modules in the spin connection (and, thereby, also in the affine connection) remain undetermined by the compatibility condition (3.20). This question is equivalent to asking which modules appear in the spin connection, but not in the torsion. It turns out that the undetermined part of the connection remains the same one as in the bosonic case, the irreducible hooks,  $\begin{smallmatrix} \square \\ \square \end{smallmatrix}$ , under the two  $so(d)$ 's, which reside entirely in  $\Omega_{a,\dot{b}\dot{c}}$ . The connection components of dimension  $\frac{1}{2}$

and  $\frac{3}{2}$ ,  $\Omega_\alpha$  and  $\Omega_{\bar{\alpha}}$ , can be completely determined when torsion is specified. We will come back to this correspondence when dealing with conventional constraints in section 3.4.

In the following, the procedure of “ordinary” supergeometry will be mimicked. This means that one needs to understand to what extent conventional transformations and constraints can be used to eliminate parts of the torsion. Then, after choosing the dimension zero torsion to a constant invariant tensor (gamma matrices), we would like to investigate the consequences of the torsion Bianchi identities.

### 3.4 Conventional constraints, Bianchi identities and field equations

The purpose of the present subsection is to investigate, using methods mimicking the ones used in ordinary supergeometry, how to get rid of the large amount of fields *a priori* present in the geometric superfields of section 3.3. This is done in two steps, first so called conventional constraints are imposed, and then the actual physical constraints. The difference is explained below. Then the Bianchi identities for the constrained torsion will be examined. The main point is to demonstrate how the physical fields are accommodated in the supergeometry. The treatment is not complete, but hopefully nevertheless convincing, especially considering the observations in section 4.

Superfields typically have too many components, and some have to be defined away by constraints. When one is dealing with a gauge theory or geometry on superspace, there are in addition a collection of different superfields in different representations (e.g. components of a super-vielbein). Typically, only the lowest-dimensional field survives as independent, and effectively contain all the higher-dimensional ones. In a geometric situation, which is based on a super-vielbein and a spin connection, one also needs to get rid of independent degrees of freedom in the spin connection.

The systematic way to deal with these constraints, and in particular to make sure that they are consistent, is the method of conventional constraints. The consistency is ensured by the derivation of the constraints from the use of certain transformations either redefining the vielbein by some matrix as

$$E_{\mathcal{M}}^{\mathcal{A}} \rightarrow E_{\mathcal{M}}^{\mathcal{B}} M_{\mathcal{B}}^{\mathcal{A}}, \quad (3.26)$$

or shifting the connection by some amount

$$\delta \Omega_{\mathcal{M}\mathcal{A}}^{\mathcal{B}} = E_{\mathcal{M}}^{\mathcal{C}} \Delta_{\mathcal{C}\mathcal{A}}^{\mathcal{B}}. \quad (3.27)$$

In order to implement the constraints covariantly, one examines how the transformations affect the torsion (so that the compatibility equation remains to hold), and which torsion components can be set to zero or a fixed value by this procedure. The principles are described in detail in ref. [48].

The maybe simplest and most clear-cut example of a conventional constraint is the constraint in  $D = 10$  super-Yang-Mills theory, formulated as a gauge theory on superspace. The rôle of the vielbein in supergeometry is then subsumed by the gauge potential on superspace, and that of the torsion by the field strength. The superfield  $A_a$  of dimension 1 may be eliminated as an independent superfield by a conventional constraint on the

field strength  $F_{\alpha\beta}$  of the same dimension. so that it becomes expressed in terms of the lowest-dimensional superfield  $A_\alpha$  of dimension  $\frac{1}{2}$ . The covariant conventional constraint is  $\gamma_a^{\alpha\beta} F_{\alpha\beta} = 0$ , where  $F = dA + A \wedge A$ .

Physical constraints, on the other hand, are the constraints that finally imply the equations of motion (in a maximally supersymmetric situation). Unlike the conventional constraints, they are not obtained through the application of transformations. Since they are implied in terms of the torsion (in order to be covariant), it is no longer guaranteed that the Bianchi identities, encoding the integrability in terms of a vielbein, are satisfied. The consequences of the physical constraints are examined by checking the torsion Bianchi identities. In a maximally supersymmetric model, this will typically imply the field equations.

In the SYM case, the physical constraint takes the form  $\gamma_{a_1 \dots a_5}^{\alpha\beta} F_{\alpha\beta} = 0$ . The conventional and physical constraints together, in this case, are  $F_{\alpha\beta} = 0$ . When the consequences of the constraints are examined, this is done by spelling out the Bianchi identity for  $F$ , which, when the constraints are imposed on field strengths, are no longer automatically satisfied. This is done by increasing dimension, corresponding to examining the consequences of the constraints in an expansion in  $\theta$ .

We now specialise to the double supergeometry corresponding to type II supergravity. The relevant orthosymplectic structure group is  $\text{OSp}(10, 10|64)$  and the locally realised subgroup  $\text{Spin}(1, 9) \times \text{Spin}(1, 9)$ . If the independent component superfields of the vielbein, the connection and the torsion are listed by dimension (the vielbein is given through a left-invariant variation  $\mathcal{E} = E^{-1} \delta E \in \text{osp}(10, 10|64)$  in order to have flat indices and to encode the group-valued property), one gets the superfields in table 1. When indices are lowered on  $\mathcal{E}$ , they are graded antisymmetric. The torsion Bianchi identities (3.22) will now be examined for the doubled formulation of type II supergravity, starting from the lowest dimension and working up. To this purpose, the graded antisymmetric module of eq. (3.22) must be decomposed into modules of  $\text{Spin}(1, 9) \times \text{Spin}(1, 9)$ , which are denoted by standard highest weight Dynkin labels. The treatment will not be complete, in that we will not examine all the irreducible modules of torsion and Bianchi identities (which is quite complicated even in ordinary supergravity). Some guidance will be taken from the possible physical fields that may or may not appear, and in this sense a complete proof of the (local) equivalence with is not provided, but hopefully nevertheless obvious.

First, note that there is no torsion below dimension  $-\frac{1}{2}$ , and thus no conventional constraint that can remove the vielbein at dimension  $-1$ . A linearised field  $\mathcal{E}_{(\alpha\beta)}$  will indeed be the superfield containing all fields (see section 4). It can be understood as the potential superfield  $B_{\alpha\beta}$ .

At dimension  $-\frac{1}{2}$  there is completely symmetric torsion  $T_{\alpha\beta\gamma}$  (which is absent in ordinary supergeometry, since the third index comes from lowering an index  $\bar{\alpha}$ ). Remember that the index  $\alpha$  downstairs collectively describes the two spinors of type II,

$$\alpha \leftrightarrow (\mathbf{16}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{16}) = (00010)(00000) \oplus (00000)(00010), \quad (3.28)$$

of  $\text{Spin}(1, 9) \times \text{Spin}(1, 9)$ , while an index  $\dot{a}$  describes a double vector,

$$\dot{a} \leftrightarrow (\mathbf{10}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{10}) = (10000)(00000) \oplus (00000)(10000). \quad (3.29)$$

dim	-1	$\mathcal{E}_\alpha^{\bar{\beta}}$		
	$-\frac{1}{2}$	$\mathcal{E}_\alpha^{\dot{b}} \sim \mathcal{E}_{\dot{a}}^{\bar{\beta}}$		$T_{\alpha\beta\gamma}$
	0	$\mathcal{E}_{\dot{a}}^{\dot{b}}, \mathcal{E}_\alpha^{\beta} \sim \mathcal{E}_{\bar{\alpha}}^{\bar{\beta}}$		$T_{\alpha\beta\dot{c}}$
	$\frac{1}{2}$	$\mathcal{E}_{\bar{\alpha}}^{\dot{b}} \sim \mathcal{E}_{\dot{a}}^{\beta}$	$\Omega_\alpha$	$T_{\alpha\beta\bar{\gamma}}, T_{\alpha\dot{b}\dot{c}}$
	1	$\mathcal{E}_{\bar{\alpha}}^{\beta}$	$\Omega_{\dot{a}}$	$T_{\dot{a}\dot{b}\dot{c}}, T_{\alpha\dot{b}\bar{\gamma}}$
	$\frac{3}{2}$		$\Omega_{\bar{\alpha}}$	$T_{\alpha\bar{\beta}\bar{\gamma}}, T_{\dot{a}\dot{b}\bar{\gamma}}$
	2			$T_{\dot{a}\bar{\beta}\bar{\gamma}}$
	$\frac{5}{2}$			$T_{\bar{\alpha}\bar{\beta}\bar{\gamma}}$

**Table 1.** Vielbein, spin connection and torsion superfields.

Conventional constraints may be imposed corresponding to the transformations of type (3.26) corresponding to  $\mathcal{E}_{\alpha\dot{b}}$ . It now turns out that not all torsion can be removed by conventional constraints. Listing the irreducible module in the torsion and conventional transformations at this dimension, they are

$$\begin{aligned}
 T_{\alpha\beta\gamma} : & \quad (00030)(00000) \oplus (00000)(00030) \oplus (00020)(00010) \oplus (00010)(00020) \\
 & \quad \oplus (10010)(00000) \oplus (00000)(10010) \oplus (10000)(00010) \oplus (00010)(10000), \\
 \mathcal{E}_{\alpha\dot{b}} : & \quad (10010)(00000) \oplus (00000)(10010) \oplus (10000)(00010) \oplus (00010)(10000) \\
 & \quad \oplus (00001)(00000) \oplus (00000)(00001).
 \end{aligned} \tag{3.30}$$

The remaining torsion, i.e., the part that can not be set to zero using a conventional constraint, is in the module

$$T_{(-\frac{1}{2}), \text{phys.}} = (00000)(00030) \oplus (00010)(00020) \oplus (00020)(00010) \oplus (00030)(00000). \tag{3.31}$$

This should be recognised as the modules occurring in the expansion of a scalar field depending on a pair of pure spinors  $\lambda^\alpha = (\lambda^{\alpha'}, \lambda^{\alpha''})$ , one for each  $\text{Spin}(1, 9)$ , to third order (remember that the expansion of a pure spinor  $\lambda$  in (00001) only contains a single module (0000 $p$ ) at order  $\lambda^p$ ). Setting the corresponding torsion components to 0 will be a physical, not conventional, constraint, and will give rise to the cohomology described in section 4. For now, we will thus proceed by setting the entire torsion at dimension  $-\frac{1}{2}$  to 0. The physical constraint, in the modules of eq. (3.31) is the only one that needs to be imposed. It will propagate through the superfield and force the theory on shell.

At dimension 0, the torsion is  $T_{\alpha\beta\dot{c}}$  and there are conventional constraints corresponding to  $\mathcal{E}_{\dot{a}\dot{b}}$  and  $\mathcal{E}_{\alpha\bar{\beta}}$ . A large number of components remain in the torsion after the conventional constraints are exhausted. There is however a Bianchi identity at dimension 0, with the index structure  $(\alpha\beta\gamma\delta)$ , where no curvature participates. This gives a number of additional constraints on the torsion. A naïve counting “torsion minus vielbein minus Bianchi” indicates that no torsion survives. It is consistent with the Bianchi identity, thanks to the



usual 10-dimensional Fierz identity  $\gamma_{a'(\alpha'\beta'}\gamma^{a'}_{\gamma'\delta')} = 0$ , to set the torsion to

$$\begin{aligned} T_{\alpha'\beta'}{}^{c'} &= 2\gamma_{\alpha'\beta'}^{c'} , \\ T_{\alpha''\beta''}{}^{c''} &= 2\gamma_{\alpha''\beta''}^{c''} , \end{aligned} \quad (3.32)$$

for which the shorthand  $T_{\alpha\beta}{}^{\dot{c}} = 2\gamma_{\alpha\beta}{}^{\dot{c}}$  is introduced. A non-zero torsion at dimension 0 is needed both to establish a relation to conventional superspace and to remove torsion through the vielbein part of conventional constraints at all dimensions. Although in ordinary (maximal) supergeometry the physical constraints appear in the torsion at dimension 0 — and here they are already accounted for at dimension  $-\frac{1}{2}$  — the appearance of non-zero invariants at dimension 0 in the form of gamma matrices is completely analogous.

Note that what has occurred in the analysis this far is quite different from ordinary superspace. There, the lowest-dimensional part of the vielbein, containing all physical fields, comes at dimension  $-\frac{1}{2}$ , and physical constraints are implemented in the torsion at dimension 0. This difference will be given a natural interpretation in section 4.

At dimension  $\frac{1}{2}$ , the fields that could possibly appear in the torsion are spinors. In supersymmetric double field theory, this is however not expected. There are vector-spinor gravitini in  $(\mathbf{10}, \mathbf{\overline{16}}) \oplus (\mathbf{\overline{16}}, \mathbf{10})$  and spinors in  $(\mathbf{1}, \mathbf{16}) \oplus (\mathbf{16}, \mathbf{1})$ . However, the “generalised gravitino” consists of both together, in the sense that also the spinors transform inhomogeneously (with a derivative on the parameter) under local supersymmetry [14, 17]. So none of these fields are allowed to appear in torsion, which is covariant. We will not show that all torsion remaining after conventional constraints are imposed is forced to vanish due to the Bianchi identities, but this is expected. As an example we will take the spinors, argued above to vanish for physical reasons. There are 3 spinors (under each  $\text{Spin}(1, 9)$ ) in  $T_{\alpha\beta\bar{\gamma}}$  and one in  $T_{\alpha\dot{b}\dot{c}}$ . The latter can be set to zero using the conventional constraint corresponding to  $\Omega_\alpha$ . The Bianchi identity (there is no curvature at this dimension) contains two spinors, and it can be checked that nothing survives that can not be absorbed by a conventional constraint, which agrees with the expectations. Concerning the rest of the modules appearing the the dimension  $\frac{1}{2}$  torsion, a detailed analysis has not been performed, but counting indicates that conventional constraints and Bianchi identities remove everything. It was noted earlier that the only part of the connection that remains undetermined, using the vielbein compatibility and fixing the torsion, is the same as in bosonic double geometry. Having vanishing torsion at dimension  $\frac{1}{2}$  completely determines  $\Omega_\alpha$  in terms of the vielbein.

Moving to dimension 1, this is where, in conventional supergeometry, field strengths for tensor fields typically appear in the torsion (and curvature). The Ramond-Ramond field strengths form a  $\text{Spin}(d, d)$  spinor, which in flat indices becomes a field  $F^{\alpha'\beta''}$  (after self-duality is imposed). Here, unlike in ordinary superspace, the dimension 1 vielbein can be used to absorb the Ramond-Ramond field strength, which becomes geometric (this is also observed in refs. [2, 3]). Effectively, by the conventional constraint, the separate superfield in the dimension 1 vielbein is eliminated, since the corresponding degrees of freedom already occur in the  $\theta$  expansion of the dimension  $-1$  vielbein.

At dimension  $\frac{3}{2}$ , there are also more conventional constraints available than usual, and the gravitino field strength in can be removed from the torsion by invoking the conventional



constraints corresponding to the spin connection  $\Omega_{\tilde{a}}$ . (Its integrability should then instead appear as a Bianchi identity  $R_{[\tilde{a}\tilde{b}\tilde{c}]\tilde{d}} = 0$  at dimension  $\frac{5}{2}$ .)

At dimension 2, one should find the equations of motion for the double geometry, as well as the equations of motion (equivalently, Bianchi identities) for the RR field strength. We have not performed the complete calculation at dimension 2, but expect the Bianchi identities for the (vanishing) torsion to contain all information.

Consider for example the Ramond-Ramond equations of motion, which may appear in the modules  $(00001)(00010) \oplus (00010)(00001)$ . This may enter the torsion as

$$T_{a'}^{\alpha'\alpha''} = \gamma_{a'}^{\alpha'\beta'} K_{\beta'}^{\alpha''} \quad (3.33)$$

( $K$  being some field in  $(00010)(00001)$ ), together with the corresponding expression with the  $\text{Spin}(1, 9)$ 's exchanged. It is forced to zero by the Bianchi identity with indices  $a''b'\gamma''\bar{\delta}'$ . Then, the Bianchi identity with indices  $a'b'\gamma'\bar{\delta}''$  ensures that it also vanishes in the curvature.

One may also check for the equations of motion of the double geometry, in the linearised coset representative  $(10000)(10000)$ . It is of course not implied by the purely bosonic Bianchi identity,  $R_{[\tilde{a}\tilde{b}\tilde{c}\tilde{d}]} = 0$ , but comes in  $R_{a'b'd''}^{c'} = \delta_{[a'}^{c'} S_{b']d''}$  and in  $R_{a''b''d'}^{c''}$ . Now there is also a Bianchi identity with structure  $\dot{a}\dot{b}\gamma\bar{\delta}$ . The part of this curvature where the spinor pair of indices, but not the vector pair, is  $so(1, 9) \oplus so(1, 9)$ -valued is identified with the corresponding part of the curvature with vector indices above,

$$R_{a'b''\gamma'}^{\delta'} = \frac{1}{4}(\gamma^{c'd'})_{\gamma'}^{\delta'} R_{a'b''c'd'} = -\frac{1}{4}(\gamma_{a'}^{c'})_{\gamma'}^{\delta'} S_{c'b''} \quad (3.34)$$

etc. (in the first step, we have used the fact that the pair of spinor indices has to take values in  $so(d)$ , since the vector indices belong to modules of the two different  $so(d)$ 's). The torsion may also contain a term  $T_{a''}^{\beta'\gamma'} \sim \gamma^{e'\beta'\gamma'} S_{e'a''}$ . It contributes to the Bianchi identity with indices  $a'b''\gamma'\bar{\delta}'$  with a term proportional to the curvature in eq. (3.34), but also with a term containing  $\delta_{\gamma'}^{\delta'} S_{a'b''}$ . This term must thus vanish both in torsion and curvature. This means that equations of motion for the generalised vielbein will be satisfied.

Torsion exists *a priori* up to dimension  $\frac{5}{2}$ . We would of course like also the higher torsion components to vanish, in order not to produce any fields outside the supergravity. The Bianchi identities seem to ensure this (the detailed identities for all modules have not been examined, but a counting supports the claim).

The results of the geometric analysis here are supported by the examination of the cohomology of the lowest-dimensional vielbein component performed in section 4.

## 4 Fields from pure spinor cohomology

The full implementation of the conventional constraints implies that all geometric superfields can be expressed in terms of the lowest-dimensional part of the vielbein with dimension  $-1$ . In section 3.4 it was noted that already the lowest-dimension Bianchi identity, at dimension  $-\frac{1}{2}$ , involving the torsion  $T_{\alpha\beta\gamma}$ , could not be completely eliminated, unless some

physical constraints (3.31) were imposed, which happen to coincide with the expansion to third order of a scalar field depending on a pair of pure Spin(1, 9) spinors  $\Lambda = (\lambda', \lambda'')$ .

This seems to indicate that the double supergeometry may be encoded in the framework of pure spinor superfields [49–58], see ref. [59] for an overview. In order to examine this hypothesis, it is convenient to work at a linearised level. The linearised dimension  $-1$  vielbein  $\mathcal{E}_{\alpha\beta}$  should appear at second order in the expansion in  $\Lambda$ . A BRST operator is

$$\mathcal{Q} = Q' + Q'' = \lambda^{\alpha'} D_{\alpha'} + \lambda^{\alpha''} D_{\alpha''}. \quad (4.1)$$

The field content is obtained from the zero-mode cohomology, which will be the tensor product of the zero-mode cohomologies of  $Q'$  and  $Q''$ . It is well known that each of them contain a  $D = 10$  super-Yang-Mills cohomology, including ghosts and antifields. The SYM cohomology is listed in table 2. The double cohomology is then obtained as the tensor products of two SYM cohomologies (labelled by modules in the two Spin(1, 9) groups). It is listed in table 3. Both tables are organised so that each column contains the  $\theta$  expansion of the superfield occurring at a certain power of the pure spinors. The superfields have been shifted down in order to display components of the same dimension on the same row. A “•” indicates the absence of zero-mode cohomology.

Let us take a closer look at the contents of table 3. Its lowest component is a scalar with ghost number 2. It represents the singlet reducibility of the super-diffeomorphisms (which, in turn, is inherited from the  $B$  field). At ghost number 1, one finds the vector and the two spinors of the super-diffeomorphisms, decomposed in Spin(1, 9)  $\times$  Spin(1, 9) modules. At ghost number 0, the physical fields appear. At dimension 0, there is the linearised coset module of the double geometry. At dimension  $\frac{1}{2}$ , one finds the correct modules (spinor and vector-spinor) for the generalised gravitino potential. At dimension 1, there is a vector, the dilaton field strength, together with a bispinor, the Ramond-Ramond field strength. It should be noted that the RR fields (as usual) only enter the supergeometry through their field strength. There is no way of accommodating the potentials together with their gauge symmetry in a geometric framework, where the gauge symmetry consists of super-diffeomorphisms. A full treatment of RR fields and gauge transformations will be given in the following section. Continuing to ghost number  $-1$ , the linearised equations of motion are read off, and one finds the correct modules at dimension  $\frac{3}{2}$  and 2. There is however a doubling, due to the section condition. Roughly speaking,  $(\partial')^2$  and  $(\partial'')^2$  contribute independently, and their sum and difference give the section condition and the equations of motion. The details of this are left for possible future examination.

We conclude that the double pure spinor cohomology represents the physical fields of the double supergeometry. It should be stressed that the appearance of two pure spinors is due to type II, and does not represent the doubling of the fermionic directions. This interpretation is so far only at linearised level around e.g. a flat superspace, and it is not geometric, since all vielbein components except the one at dimension  $-1$  have been discarded. Note that the zero-mode cohomology of table 3 (unlike the one in table 2, and e.g. the cohomology of  $D = 11$  supergravity [57, 58]) does not exhibit a field-antifield symmetry. This is normal in a situation including self-dual fields, and prevents the existence of a pure

ghost# =	1	0	-1	-2	-3
dim = 0 (00000)					
$\frac{1}{2}$	•	•			
1	•	(10000)	•		
$\frac{3}{2}$	•	(00001)	•	•	
2	•	•	•	•	•
$\frac{5}{2}$	•	•	(00010)	•	•
3	•	•	(10000)	•	•
$\frac{7}{2}$	•	•	•	•	•
4	•	•	•	(00000)	•
$\frac{9}{2}$	•	•	•	•	•

**Table 2.** The zero-mode cohomology of  $D = 10$  super-Yang-Mills.

spinor superfield action. In principle, interactions may be introduced as deformations of the cohomology, but there is no full Batalin-Vilkovisky pure spinor superfield framework.

## 5 OSp spinors and Ramond-Ramond fields

In this section, a basis for the the  $\text{OSp}(d, d|2s)$  vector representation is used such that  $dZ^{\mathcal{M}} = (dx^m, d\tilde{x}_m; d\theta^\mu, d\tilde{\theta}_\mu)$ . Here,  $M = (m, \mu)$  is a  $\text{GL}(d|s)$  index, and the invariant metric takes the form (3.2).

### 5.1 OSp spinors

In complete analogy with  $\text{Spin}(d, d)$ , where a spinor is realised as a sum of even or odd forms in  $d$  dimensions, we now form a basis for the infinite-dimensional spinor representations of  $\text{osp}(d, d|2s)$  as consisting of all superforms

$$e^{m_1 \dots m_p | \mu_1 \dots \mu_q} = dx^{m_1} \wedge \dots \wedge dx^{m_p} \wedge d\theta^{\mu_1} \wedge \dots \wedge d\theta^{\mu_q}. \quad (5.1)$$

Chiral spinors have  $p + q$  even or odd. Ramond-Ramond superfields are even or odd forms on ordinary superspace. Fields in the two representations are

$$\begin{aligned} S &= \bigoplus_{P \in 2\mathbb{N}} \frac{p!}{d} z^{M_1} \wedge \dots \wedge dz^{M_P} S_{M_P \dots M_1}, \\ C &= \bigoplus_{P \in 2\mathbb{N}+1} \frac{p!}{d} z^{M_1} \wedge \dots \wedge dz^{M_P} C_{M_P \dots M_1}. \end{aligned} \quad (5.2)$$

Here, the conventions for ordering and contractions of section 3.1 lead to the standard conventions for forms on superspace.

gh# =	2	1	0	-1	-2	-3	-4
dim = -2(00000)(00000)							
$-\frac{3}{2}$	•	•					
-1	•	(00000)(10000)	•				
		(10000)(00000)					
$-\frac{1}{2}$	•	(00000)(00001)	•	•			
		(00001)(00000)					
0	•	•	(10000)(10000)	•	•		
			(00000)(00010)				
$\frac{1}{2}$	•	•	(00001)(10000)		•	•	•
			(00010)(00000)				
			(10000)(00001)				
			(00000)(10000)				
1	•	•	(00001)(00001)	•	•	•	•
			(10000)(00000)				
$\frac{3}{2}$	•	•	•	(00010)(10000)	•	•	•
				(10000)(00010)			
				2(00000)(00000)			
2	•	•	•	(00001)(00010)	•	•	•
				(00010)(00001)			
				2(10000)(10000)			
$\frac{5}{2}$	•	•	•	(00001)(10000)	•	•	•
				(10000)(00001)			
3	•	•	•	•	(00000)(10000)	•	•
					(00010)(00010)		
					(10000)(00000)		
					(00000)(00001)		
$\frac{7}{2}$	•	•	•	•	(00010)(10000)	•	•
					(00001)(00000)		
					(10000)(00010)		
4	•	•	•	•	(10000)(10000)	•	•
$\frac{9}{2}$	•	•	•	•	•	(00000)(00010)	•
						(00010)(00000)	
5	•	•	•	•	•	(00000)(10000)	•
						(10000)(00000)	
$\frac{11}{2}$	•	•	•	•	•	•	•
6	•	•	•	•	•	•	(00000)(00000)

**Table 3.** Zero-mode cohomology of supersymmetric double field theory as (SYM)<sup>2</sup>.

“Super-Gamma matrices”  $\Sigma^{\mathcal{M}}$  are now introduced through their action by wedge products and contractions on the “spinor” superforms:

$$\begin{aligned} \Sigma^M \omega = \sqrt{2} dz^M \wedge \omega & : \begin{cases} \Sigma^m \omega = \sqrt{2} dx^m \wedge \omega \\ \Sigma^\mu \omega = \sqrt{2} d\theta^\mu \wedge \omega \end{cases} \\ \Sigma_M \omega = \sqrt{2} \iota_M \omega & : \begin{cases} \Sigma_m \omega = \sqrt{2} \iota_m \omega \\ \Sigma_\mu \omega = \sqrt{2} \iota_\mu \omega \end{cases} \end{aligned} \quad (5.3)$$

This definition immediately leads to

$$\{\Sigma^{\mathcal{M}}, \Sigma^{\mathcal{N}}\} = 2\hat{H}^{\mathcal{MN}} \mathbb{1}, \quad (5.4)$$

where  $\{A, B\} = AB + (-1)^{|A||B|}BA$  is the graded anticommutator. Orthosymplectic transformations are realised as

$$\delta_f \omega = -\frac{1}{4} \Sigma^{\mathcal{MN}} f_{\mathcal{N}\mathcal{M}} \omega, \quad (5.5)$$

with  $\Sigma^{\mathcal{MN}} = \Sigma^{[\mathcal{M}} \Sigma^{\mathcal{N}]}$  graded antisymmetric.

The two spinor chiralities form infinite-dimensional highest weight modules  $S$  and  $C$  of (the double cover of)  $\text{OSp}(d, d|2s)$ . Their conjugates modules are lowest weight, and the only singlets appearing in tensor products of spinors and cospinors are in  $S \otimes \bar{S}$  and  $C \otimes \bar{C}$  (unlike the  $\text{Spin}(d, d)$  situation where the conjugate module  $\bar{S}$  is  $S$  or  $C$ , depending on dimension). Note that action with a single  $\Sigma$  matrix changes the chirality of the spinor.

## 5.2 Ramond-Ramond superfields

The treatment of Ramond-Ramond fields as dynamical spinors in double field theory was introduced in ref. [19]. Here, it is extended to superspace. In the supergeometry, only the RR field strengths appeared, since their gauge symmetry could not be encoded. This will now be remedied.

Introduce a Dirac operator  $\not{\partial} = \Sigma^{\mathcal{M}} \partial_{\mathcal{M}}$ , mapping  $S$  to  $C$  and vice versa. It becomes nilpotent thanks to the section condition (3.13):

$$\not{\partial}^2 \omega = \Sigma^{\mathcal{M}} \partial_{\mathcal{M}} \Sigma^{\mathcal{N}} \partial_{\mathcal{N}} \omega = \Sigma^{\mathcal{M}} \Sigma^{\mathcal{N}} \partial_{\mathcal{N}} \partial_{\mathcal{M}} \omega = 2\hat{H}^{\mathcal{MN}} \partial_{\mathcal{N}} \partial_{\mathcal{M}} \omega = 0. \quad (5.6)$$

It is therefore meaningful to let a gauge field (RR superpotential) transform in  $S$ , say. Let us call this field  $S$ . It will have a field strength  $F = \not{\partial} S$  which is invariant under the gauge transformations<sup>7</sup>  $\delta S = \not{\partial} \Lambda$ .

We must however check if the Dirac operator is covariant. This is done by replacing the naked derivative with a covariant one,  $\not{\partial} \rightarrow \not{D}$ , containing the affine super-connection  $\Gamma$ . One can also allow for a weight  $w$ .

$$\begin{aligned} \not{D} \omega &= \Sigma^{\mathcal{M}} D_{\mathcal{M}} \omega = \Sigma^{\mathcal{M}} \left( \partial_{\mathcal{M}} - \frac{1}{4} \Gamma^{\mathcal{MN}\mathcal{P}} \Sigma_{\mathcal{P}\mathcal{N}} - w \Gamma^{\mathcal{N}}_{\mathcal{N}\mathcal{M}} \right) \omega \\ &= \left( \not{\partial} - \frac{1}{4} \Gamma^{\mathcal{MN}\mathcal{P}} \Sigma_{\mathcal{P}\mathcal{N}} - \left( w - \frac{1}{2} \right) \Gamma^{\mathcal{N}}_{\mathcal{N}\mathcal{M}} \Sigma_{\mathcal{M}} \right) \omega. \end{aligned} \quad (5.7)$$

<sup>7</sup>These gauge transformations are infinitely reducible, with spinors all the way down. This happens already for the  $\text{Spin}(d, d)$  RR-fields in double geometry. There, the naive counting  $1 - 1 + 1 - 1 + \dots = \frac{1}{2}$  gives the correct counting of the off-shell degrees of freedom modulo gauge transformations.

As for the  $\text{Spin}(d, d)$  spinor, the  $\Sigma^{(3)}$  term is torsion, and the (naked) Dirac operator becomes covariant if  $w = \frac{1}{2}$ , in which case it becomes

$$\not{D}\omega = \left( \not{D} - \frac{1}{6} T^{\mathcal{AB}\mathcal{C}} \Sigma_{\mathcal{C}\mathcal{B}\mathcal{A}} \right) \omega. \quad (5.8)$$

In a suitable basis adjusted to the solution of the section condition, it is the exterior derivative on superspace forms. Take only  $T_{\alpha\beta}{}^c = 2\gamma_{\alpha\beta}{}^c$  non-vanishing. In a flat basis the  $\text{OSp}$  spinor is a form spanned by  $E^a$  and  $E^\alpha$ . The torsion term in eq. (5.8) becomes  $-E^\beta \wedge E^\alpha \gamma_{\alpha\beta}{}^c \iota_c$ , corresponding to the standard dimension 0 torsion on ordinary superspace. This leads to the usual dimension 0 Chevalley-Eilenberg cohomologies [60] for the Ramond-Ramond superfields, listed for example in ref. [61].

It should be expected that the Ramond-Ramond double superfield encodes also the supergeometric data, i.e., all the fields in double supergeometry, discussed in section 3. This would be in accordance with e.g.  $D = 11$  supergravity, where the true basic superfield is the one corresponding to the 3-form tensor [57, 58]. We have not checked how this happens, but leave it for future work. Due to the infinite reducibility of the gauge transformation, it is not clear what kind of structure will replace the pure spinor superfields of section 4. Whether such a structure can be used for a Batalin-Vilkovisky action (or pseudo-action, remembering the selfduality), or some similar efficient description of the full dynamics, remains to be seen.

### 5.3 Pure $\text{OSp}$ spinors and super-sections

Just as pure spinors define isotropic subspaces — sections — in double field theory, a pure  $\text{OSp}$  spinor defines a super-section. This is an isotropic embedding of an ordinary  $(d|s)$ -dimensional superspace in the  $(2d|2s)$ -dimensional doubled superspace. Since the spinors are infinite-dimensional, there is no analogue of a Mukai pairing, and it is more convenient to use the traditional definition of a pure spinor due to Cartan [62] than to form spinor bilinears.

A pure spinor is a spinor that is annihilated by a maximal isotropic set of  $\Sigma$ -matrices. Inspecting eq. (5.3), it is clear that the spinor represented by 1 has this property; it is annihilated by  $\Sigma_M = \iota_M$ . Such a pure spinor lies in a minimal (finite-dimensional) orbit of the (infinite-dimensional)  $S$  module under  $\text{OSp}(d, d|2s)$ . Acting with the supergroup generates the orbit

$$\Lambda = e^{\Phi+B\wedge} 1, \quad (5.9)$$

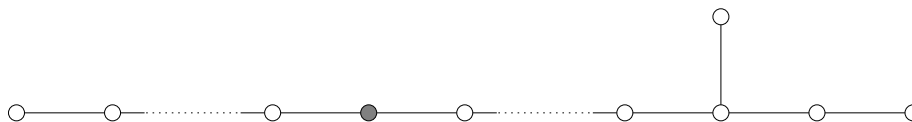
where  $B$  is a super-2-form in  $(d|s)$  dimensions. The pure spinor space is the supergroup quotient

$$\Pi = \frac{\text{OSp}(d, d|2s)}{\text{GL}(d|s) \ltimes \mathcal{B}} \times \mathbb{R}, \quad (5.10)$$

where  $\mathcal{B}$  is the graded antisymmetric module. The dimensionality of pure spinor space is

$$\dim(\Pi) = \left( \frac{d(d-1)}{2} + \frac{s(s+1)}{2} + 1 \mid ds \right), \quad (5.11)$$

which clearly matches the dimensionality of the orbit in eq. (5.9).



**Figure 2.** The Dynkin diagram for a superalgebra underlying exceptional supergeometry?

The pure OSp spinors should be relevant for the formulation of D-brane dynamics in double superspace, much in the same way pure spinors enter the construction of D-branes in (bosonic) double field theory [63–65]. The D-brane, like the section, is a maximal isotropic subspace [8].

## 6 Conclusions

We have constructed a double supergeometry, where covariance under super-diffeomorphisms is manifest. Ordinary superspace is obtained as a super-section, just as ordinary space is a section in double geometry. In a maximally supersymmetric situation, the fields will be on shell, when a set of physical constraints (in contrast to the conventional ones) are imposed at the lowest-dimensional torsion. The structure is reflected in the product of two super-Yang-Mills pure spinor cohomologies.

A few comments on the relation to the work by Hatsuda et al. [2, 3] are in place. That work starts from affine super-Poincaré algebras for left- and right-movers on a string, which leads to an orthosymplectic group of significantly higher dimension than the one in the present paper. Then  $\kappa$ -symmetry and Virasoro symmetry are imposed in order to restrict the background. There is only torsion, no curvature, but what in the present work is curvature is encoded as part of the torsion (since what we here call spin connection is made part of a big vielbein). A treatment of super-diffeomorphisms is not performed. In many respects, the results concerning supergravity fields of the present work and of refs. [2, 3], such as the appearance of a “prepotential”, the vielbein at dimension  $-1$ , seem to be consistent, and it is likely that they are equivalent.

A natural question is how this can be continued to exceptional field theory. Unlike the orthogonal group  $O(d, d)$ , the exceptional duality groups  $E_{n(n)}$ , have no finite-dimensional super-extensions [66]. The present work may offer some clues. On one hand it is well known that the coordinate module in exceptional geometry contains a spinor when the U-duality group  $E_{n(n)}$  is reduced to the T-duality group  $\text{Spin}(n-1, n-1)$ . On the other hand we have seen in section 5 that superspace counterpart of the T-duality spinor is an infinite-dimensional module. It seems clear that it will be necessary to start from a superspace with an infinite-dimensional coordinate module, and since there are no finite-dimensional superalgebras at hand, also the super-extension of the U-duality group should be infinite-dimensional, maybe of the type depicted in figure 2. Examination of this hypothesis will be the subject of future work.

## Acknowledgments

The author would like to extend his gratitude to Machiko Hatsuda for discussions and explanation of her work, and to Jakob Palmkvist for clarifying some aspects of superalgebras.

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